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Your Roll No.....

Sr. No. of Question Paper : 1223

F

Unique Paper Code : 2352011202

Name of the Paper : CALCULUS

Name of the Course : B.Sc. (H) Mathematics
UGCF-2022

Semester : II – DSC 5

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **all** questions by selecting **three** parts from each question.
3. **All** questions carry equal marks.
4. Use of Calculator is not allowed.

1. (a) State and prove the sequential criterion for the limit of a real valued function. (5)

(b) Use $\epsilon - \delta$ definition of limit to establish the following limit : (5)

$$\lim_{x \rightarrow 2} \frac{1}{1-x} = -1.$$

P.T.O.

(c) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined as (5)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f has a limit only at $x = 0$.

(d) Let $A \subseteq \mathbb{R}$, let $f: A \rightarrow \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A . If $\lim_{x \rightarrow c} f > 0$, then show that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$. (5)

2. (a) If f is continuous at x_0 and g is continuous at $f(x_0)$ then prove that the composite function $g \circ f$ is continuous at x_0 . (5)

(b) Let $f(x) = \frac{1}{x} \sin \frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that f is discontinuous at 0. (5)

(c) State Intermediate Value Theorem. Prove that $xe^x = 1$ for some x in $(0,1)$. (5)

(d) Let f be a continuous real-valued function with domain (a, b) . Show that if $f(r) = 0$ for each rational number r in (a, b) , then $f(x) = 0$ for all $x \in (a, b)$. (5)

3. (a) Prove that if a real valued function f is continuous on $[a, b]$ then it is uniformly continuous on $[a, b]$. (5)

- (b) Show that the function $f(x) = \frac{1}{x}$ is uniformly continuous on (a, ∞) for $a > 0$ but it is not uniformly continuous on $(0,1)$. (5)
- (c) Let $f(x) = |x| + |x - 1|$, $x \in \mathbb{R}$. Draw the graph and give the set of points where it is not differentiable. Justify also. (5)
- (d) Prove that if f and g are differentiable on \mathbb{R} , if $f(0) = g(0)$ and if $f'(x) \leq g'(x)$ for all $x \in \mathbb{R}$, then $f(x) \leq g(x)$ for $x \geq 0$. (5)
4. (a) State and prove Mean Value Theorem. (5)
- (b) State Intermediate Value Theorem for derivatives. Suppose f is differentiable on \mathbb{R} and $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.
- (i) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0,2)$.
- (ii) Show that $f'(x) = \frac{1}{7}$ for some $x \in (0,2)$. (5)
- (c) Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. (5)

- (d) Let f be defined on \mathbb{R} and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. (5)
5. (a) Let f be differentiable function on an open interval (a, b) . Then show that f is increasing on (a, b) if $f'(x) \geq 0$. (5)
- (b) If $y = e^{\tan^{-1}x}$, prove that (5)
- $$(1 + x^2)y_{n+2} + (2(n + 1)x - 1)y_{n+1} + n(n + 1)y_n = 0.$$
- (c) If $y = \cos(m \sin^{-1} x)$, find $y_n(0)$. (5)
- (d) Stating Taylor's theorem find Taylor series expansion of e^x . (5)
6. (a) Find $\lim_{x \rightarrow +\infty} \left[x - \ln(x^2 + 1) \right]$. (5)
- (b) Determine the position and nature of the double points on the curve (5)
- $$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$$
- (c) Sketch a graph of the rational function showing the horizontal, vertical and oblique asymptote (if any) of $y = \frac{x^2 - 2}{x}$. (5)
- (d) Sketch the curve in polar coordinates of $r = \sin 2\theta$. (5)